Properties of the Soliton-Lattice State in Double-Layer Quantum Hall Systems

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Application of a sufficiently strong parallel magnetic field $B_\parallel > B_c$ produces a soliton-lattice (SL) ground state in a double-layer quantum Hall system. We calculate the ground-state properties of the SL state as a function of $B_\parallel$ for total filling factor $\nu_T = 1$, and obtain the total energy, anisotropic SL stiffness, Kosterlitz-Thouless melting temperature, and SL magnetization. The SL magnetization might be experimentally measurable, and the magnetic susceptibility diverges as $|B_\parallel - B_c|^{-1}$.

Keywords: quantum Hall, solitons, magnetization, Kosterlitz-Thouless

I. INTRODUCTION

At sufficiently small layer separation, a double-layer quantum Hall (2LQH) system [1,2] is an unusual quantum itinerant ferromagnet. [3,4] The 2LQH system can be mapped to an equivalent spin-1/2 system by equating "up" ("down") pseudospins with electrons in the upper (lower) layer. [3] For any finite layer separation d, the itinerant ferromagnet has an XY symmetry, so that the orientation of a pseudospin at location $\mathbf{r}$ is specified by its angle $\theta(\mathbf{r})$ in the xy plane.

Murphy *et al.* have investigated the effect of an in-plane magnetic field $B_\parallel$ on 2LQH systems and find evidence for a phase transition between two competing QH ground states at a critical value $B_\parallel = B_c$. [5] These two ground states have been explained theoretically [3] by showing that $B_\parallel$ produces a rotating Zeeman field seen by the pseudospins. This gives a pseudospin contribution to the ground-state energy of the Pokrovsky-Talapov (PT) form, [6,7]

$$E = \int d^2r \left[\frac{\rho_s}{2}|\nabla\tilde{\theta} - \mathbf{Q}|^2 + \frac{t}{2\pi\ell^2}(1-\cos\tilde{\theta})\right], \tag{1.1}$$

where $\mathbf{Q} \equiv 2\pi\hat{\mathbf{z}} \times \mathbf{B}_\parallel d/\phi_0$ defines the parallel magnetic-field wave vector, $\tilde{\theta}(\mathbf{r}) \equiv \theta(\mathbf{r}) + \mathbf{Q}\cdot\mathbf{r}$, $t = t_0 e^{-Q^2\ell^2/4}\sqrt{4\nu_1\nu_2}$ is the tunneling energy (t_0 is the tunneling energy when $Q = 0$) [8] and $\rho_s = \rho_E(4\nu_1\nu_2)$ is the pseudospin stiffness (ρ_E is the interlayer exchange stiffness when $\nu_1 = \nu_2 = 1/2$) in the Hartree-Fock Approximation (HFA), ν_j is the filling factor of layer j, and the energy is measured relative to the ground-state energy for $B_\parallel = 0$. Note that by adjusting the front and back gate voltages of the sample, ν_1 and ν_2 may be varied (with $\nu_T \equiv \nu_1 + \nu_2 = 1$), thereby allowing t and ρ_s to be adjusted.

For small $B_\parallel$, Eq. (1.1) is minimized by $\tilde{\theta}(\mathbf{r}) = 0$. This is the commensurate (C) ground state. For all finite $B_\parallel > B_c$, the pseudospin polarization has broken translational symmetry, and a soliton-lattice (SL) state results. For large $B_\parallel$ the pseudospins behave (almost) as if $t = 0$. This work focusses on calculating the ground-state properties of the SL state, for all $B_\parallel > B_c$. Interestingly, it is not necessary to solve for the form of $\tilde{\theta}(\mathbf{r})$ in order to calculate the total energy of the system. [6,7,9]

Minimizing E with respect to $\tilde{\theta}$ gives the 2D sine-Gordon equation (SGE), $\xi^2\nabla^2\tilde{\theta} = \sin\tilde{\theta}$, where $\xi/\ell = \sqrt{2\pi\rho_s/t}$. We shall give

numerical values of our results for a hypothetical GaAs 2LQH sample with total density $1.1\times10^{11}\mathrm{cm}^{-2}$, layer separation $d=21.1$ nm, and tunneling energy $t_0=0.1$ meV. Such a sample would have $\ell\approx11.8$ nm, $d/\ell=1.8$, and $\hbar\omega_{\mathrm{c}}\approx8$ meV for $\nu_{\mathrm{T}}=1$, and $\rho_{\mathrm{E}}\approx0.08$ meV and $\xi\approx26.5$ nm in the HFA.

II. SOLITON-LATTICE STATE

We investigate solutions of the SGE of the form $\tilde\theta(\mathbf{r})=\tilde\theta\,[\hat{\mathbf{e}}_1\cdot(\mathbf{r}-\mathbf{r}_0)]$, where $\hat{\mathbf{e}}_1$ is some unit vector in the xy plane, so that $\xi^2\partial_1^2\tilde\theta=\sin\tilde\theta$, which is closely analogous to the equation of motion of a pendulum. The conserved quantity analogous to the total energy of a pendulum is $2c^2\equiv(1/2)\xi^2(\partial_1\tilde\theta)^2-(1-\cos\tilde\theta)$. Defining $\beta=\tilde\theta/2$ leads to the equation

$$\xi\partial_1\beta=\pm\sqrt{c^2+\sin^2\beta}. \tag{2.1}$$

When $c=0$, Eq. (2.1) gives $\tilde\theta_{\mathrm{ss}}(\mathbf{r})=4\arctan\exp[\hat{\mathbf{e}}_1\cdot(\mathbf{r}-\mathbf{r}_0)/\xi]$, which represents a single soliton of width ξ, corresponding to the motion of a pendulum that completes a single revolution in infinite time. The energy per unit length of a single soliton relative to the C ground-state energy may be computed from Eq. (1.1) as $E_{ss}/L_2=\rho_{\mathrm{s}}\,(8/\xi-2\pi\hat{\mathbf{e}}_1\cdot\mathbf{Q})$, where L_2 is the sample length perpendicular to $\hat{\mathbf{e}}_1$. The lowest energy solitons have $\hat{\mathbf{e}}_1=\hat{\mathbf{Q}}$ and their energy vanishes when $Q=Q_{\mathrm{c}}\equiv4/(\pi\xi)$. Thus for $Q>Q_{\mathrm{c}}$ it is energetically favorable to create solitons. The solitons are weakly repulsive, and form a SL. Analogous soliton effects also occur in long Josephson junctions. [10]

The SL spacing L_{s} may be determined by noting that over one period of the SL, β changes by π. Thus

$$L_{\mathrm{s}}=\int_{-\pi/2}^{\pi/2}\frac{d\beta}{\partial_1\beta}=2\eta K(\eta), \tag{2.2}$$

where we have used Eq. (2.1), defined $\eta\equiv1/\sqrt{c^2+1}$, and where $K(\eta)$ is the complete elliptic integral of the first kind. [11] We define the SL wave vector $\mathbf{Q}_{\mathrm{s}}\equiv(2\pi/L_{\mathrm{s}})\hat{\mathbf{e}}_1$, so that Eq. (2.2) becomes

$$Q_{\mathrm{s}}/Q_{\mathrm{c}}=(\pi/2)^2/[\eta K(\eta)]. \tag{2.3}$$

Note that $\eta\to0$ corresponds to $Q_{\mathrm{s}}\to Q\to\infty$, whereas $\eta\to1$ corresponds to the C-SL transition, where $Q\to Q_{\mathrm{c}}$ and $Q_{\mathrm{s}}\to0$.

The energy per unit area is obtained by expressing Eq. (1.1) as an integral over β [cf. Eq. (2.2)], which gives

$$\frac{E}{L_1L_2}=\frac{\rho_{\mathrm{s}}}{\xi^2}[(\frac{Q^2}{2}-\mathbf{Q}\cdot\mathbf{Q}_{\mathrm{s}})\xi^2 - 2(\frac{1}{\eta^2}-1)+Q_{\mathrm{c}}Q_{\mathrm{s}}\xi^2\frac{E(\eta)}{\eta}], \tag{2.4}$$

where $E(\eta)$ is the complete elliptic integral of the second kind [11] and L_1L_2 is the sample area. The value of $\mathbf{Q}_{\mathrm{s}}$ that minimizes the energy per unit area is found by differentiating Eq. (2.4) with respect to $\mathbf{Q}_{\mathrm{s}}$, holding $\mathbf{Q}$ constant. Using the identity [11] $dE/d\eta=[E(\eta)-K(\eta)]/\eta$, one obtains

$$Q/Q_{\mathrm{c}}=E(\bar\eta)/\bar\eta. \tag{2.5}$$

Equations (2.3) and (2.5) together determine the equilibrium SL wave vector $\bar{\mathbf{Q}}_{\mathrm{s}}(\mathbf{Q})$ that minimizes the energy. For $Q/Q_{\mathrm{c}}\to\infty$, $\bar Q_{\mathrm{s}}/Q\approx1-(1/2)[(\pi/4)(Q_{\mathrm{c}}/Q)]^4$. For $Q\to Q_{\mathrm{c}}$, $\bar Q_{\mathrm{s}}/Q_{\mathrm{c}}\sim(\pi^2/2)/\ln(1/\epsilon)$ asymptotically, [6,12] where $\epsilon\equiv Q/Q_{\mathrm{c}}-1$. If it were possible to achieve $\epsilon\sim10^{-2}$ (e.g., by gating the sample to tune Q_{c}), surface acoustic wave (SAW) techniques might detect the SL when L_{s} matched the SAW wavelength. The general SL ground-state solution for $\tilde\theta$ is given by $\sin[(\tilde\theta_{\mathrm{SL}}-\pi)/2]=\mathrm{sn}[\hat{\mathbf{e}}_1\cdot(\mathbf{r}-\mathbf{r}_0)/\eta\xi,\eta]$, where sn is the sine-amplitude Jacobian function. [10]

III. STIFFNESSES

The elastic constants of the soliton lattice are given by the stiffness tensor

$$K_{ij} \equiv \lim_{\mathbf{Q}_\mathrm{s}\to\bar{\mathbf{Q}}_\mathrm{s}} \left[\frac{\partial^2(E/L_1L_2)}{\partial Q_{\mathrm{s}i}\partial Q_{\mathrm{s}j}}\right]_{\mathbf{Q}}, \qquad (3.1)$$

which we calculate here for all $Q \geq Q_\mathrm{c}$ using Eqs. (2.4) and (2.5). Our results agree with those obtained in Ref. [12] for $Q \to Q_\mathrm{c}$.

Using the results of Sec. II, and the identity [11] $dK/d\eta = E(\eta)/[\eta(1-\eta^2)] - K(\eta)/\eta$, the compressional elastic constant K_{11} is found to be

$$\frac{K_{11}}{\rho_\mathrm{s}} = \frac{\partial Q}{\partial \bar{Q}_\mathrm{s}} \to \begin{cases} 1-\frac{3}{2}\left(\frac{\pi}{4}\frac{Q_\mathrm{c}}{Q}\right)^4, & Q/Q_\mathrm{c}\to\infty \\ (2/\pi^2)\epsilon\ln^2(1/\epsilon), & Q/Q_\mathrm{c}\to 1 \end{cases} \qquad (3.2)$$

As expected, $K_{11} \to \rho_\mathrm{s}$ in the limit $Q/Q_\mathrm{c} \to \infty$; $K_{11} \to 0$ for $Q/Q_\mathrm{c} \to 1$ because of the short-range repulsions between the solitons. The shear elastic constant K_{22} is

$$\frac{K_{22}}{\rho_\mathrm{s}} = \frac{Q}{\bar{Q}_\mathrm{s}} \to \begin{cases} 1+\frac{1}{2}\left(\frac{\pi}{4}\frac{Q_\mathrm{c}}{Q}\right)^4, & Q/Q_\mathrm{c}\to\infty \\ (2/\pi^2)\ln(1/\epsilon), & Q/Q_\mathrm{c}\to 1 \end{cases} \qquad (3.3)$$

Like K_{11}, approaches $K_{22} \to \rho_\mathrm{s}$ for $Q/Q_\mathrm{c} \to \infty$. However, K_{22} diverges for $Q \to Q_\mathrm{c}$.

The SL phase of the PT model can undergo a KT transition, [7] with a transition temperature $k_BT_\mathrm{KT} \sim (\pi/2)\sqrt{K_{11}K_{22}}$. The KT transition is probably most easily measured in devices with oppositely directed currents in each layer. [4] However, because the SL dislocations are electrically charged, the KT transition might increase the longitudinal resistivity even in devices without separately contacted layers, due to the proliferation of unbound charged dislocations above T_KT.

We now calculate the stiffness tensor $\tilde{K}_{ij}$ relevant to the sound velocities of the SL by writing $\tilde{\theta}(\mathbf{r}) = \tilde{\theta}_0(\mathbf{r}) + \delta\tilde{\theta}(\mathbf{r})$, where $\tilde{\theta}_0$ is the ground-state solution of the SGE which minimizes the PT energy (1.1) and $\delta\tilde{\theta}$ is a small deviation of $\tilde{\theta}$ from $\tilde{\theta}_0$. To quadratic order in $\tilde{\theta}$, the change in the PT energy (1.1) of the 2LQH system is

$$\delta E = \frac{1}{2}\int \frac{d^2r}{2\pi\ell^2}\delta\tilde{\theta}\left[t\cos\tilde{\theta}_0 - 2\pi\ell^2\rho_\mathrm{s}\nabla^2\right]\delta\tilde{\theta} \qquad (3.4)$$

where we have integrated by parts and made use of the SGE. The energy δE is minimized by choosing $\delta\tilde{\theta}$ from among the eigenvalues of bracketed Schrödinger-like operator in Eq. (3.4). For $\mathbf{Q} = \hat{\mathbf{x}}$, $\tilde{\theta}_0 = \tilde{\theta}_0(x)$ and we may write $\delta\tilde{\theta}(\mathbf{r}) \propto \exp(iq_yy)\delta\tilde{\theta}(x)$. When $\tilde{\theta}_0(\mathbf{r}) = \tilde{\theta}_{SL}(\mathbf{r})$, the eigenvalue equation for bracketed equation in Eq. (3.4) becomes Lamé's equation after a simple rescaling of x. [10] The relevant "vortex oscillation" solutions to Lamé's equation follow from Ref. [10] and Eqs. (2.3) and (2.5) in the long-wavelength limit: $\delta E/L_1L_2 = (1/2)(K_{11}q_x^2 + \rho_\mathrm{s}q_y^2)$, where K_{11} is equal to the compressional stiffness (3.2). Thus $\tilde{K}_{11} = K_{11}$, while $\tilde{K}_{22} = \rho_\mathrm{s}$ is independent of Q.

IV. MAGNETIZATION AND MAGNETIC SUSCEPTIBILITY

The SL makes a small but possibly measurable contribution to the magnetization of the 2LQH system. The parallel-field magnetization per unit area is

$$\mathbf{M}_\parallel \equiv -\frac{\partial}{\partial \mathbf{B}_\parallel}\left(\frac{\bar{E}}{L_1L_2}\right) = -M_0(\mathbf{Q}-\bar{\mathbf{Q}}_\mathrm{s})/Q_\mathrm{c}, \qquad (4.1)$$

where the equilibrium value of $\bar{E}/L_1L_2$, found by using Eqs. (2.3) and (2.5) in Eq. (2.4), and we have used the results of Sec. II. $M_0 \equiv 2\pi\rho_\mathrm{s}Q_\mathrm{c}d/\phi_0$ sets the scale of the magnetization density, which we plot in Fig. 1. It is useful to compare the total SL magnetization $M_0L_1L_2$ to the scale of the Landau diamagnetism in a $\nu = 1$ QH system:

$$\frac{M_0L_1L_2}{N\mu_\mathrm{B}^*} = \frac{M_0\phi_0}{\mu_\mathrm{B}^*B_\perp} = 16\frac{d}{\xi}\frac{\rho_\mathrm{s}}{\hbar\omega_\mathrm{c}} \sim 0.1, \qquad (4.2)$$

where $\mu_\mathrm{B}^* = e\hbar/2m^*c$ is the effective Bohr magneton in GaAs, and $m^* \approx 0.067m_\mathrm{e}$ is the

effective mass. This shows that in the HFA, the SL magnetization is roughly an order of magnitude smaller than the Landau diamagnetism, which has been measured in GaAs single-layer heterostructures. [13] The torque on a tilted 2LQH sample has both a Landau-diamagnetic component $\tau_\perp = M_\perp B_\parallel$ and a SL component $\tau_\parallel = M_\parallel B_\perp$. The smallness $M_\parallel$ is somewhat compensated by the size of $B_\perp$ in the expression for $\tau_\parallel$.

The parallel-field magnetic susceptibility $\chi_\parallel \equiv \partial M/\partial B_\parallel$ is given by

$$\chi_\parallel = \chi_0(\partial\bar{Q}_{\rm s}/\partial Q - 1) = \chi_0(\rho_{\rm s}/K_{11} - 1) \quad (4.3)$$
$$\to \chi_0 \begin{cases} (3/2)[(\pi/4)(Q_{\rm c}/Q)]^4, & Q/Q_{\rm c} \to \infty \\ (\pi^2/2)/[\epsilon \ln(1/\epsilon)], & Q/Q_{\rm c} \to 1 \\ -1, & Q < Q_{\rm c} \end{cases}$$

where $\chi_0 \equiv (2\pi d/\phi_0)^2 \rho_{\rm s} \sim 5 \times 10^{-14}$m. Near the C-SL transition, $\chi_\parallel \sim 1/\epsilon$, with logarithmic corrections. It might not prove practical to make AC measurements of $\chi_\parallel$ because of substantial eddy-currents in the QH regime.

V. ACKNOWLEDGMENTS

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FIG. 1. SL contribution to the magnetization density (dots), which drops precipitously for $Q > Q_{\rm c}$. The C-phase result (solid line) holds for $Q < Q_{\rm c}$.